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Discontinuous games with asymmetric information: An extension of Reny's existence theorem ☆

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ABSTRACT

We introduce asymmetric information to games with discontinuous payoffs and prove new equilibrium existence theorems. In particular, the seminal work of Reny (1999) is extended to a Bayesian preferences framework.

1. Introduction

Games with discontinuous payoffs have been used to model a variety of important economic problems; for example, Hotelling location games, Bertrand competition, and various auction models. The seminal work by Reny (1999) proposed the "better reply security" condition and proved the equilibrium existence in quasiconcave compact games with discontinuous payoffs. Since the hypotheses are sufficiently simple and easily verified, the increasing applications of his results has widened significantly in recent years, as evidenced by Jackson and Swinkels (2005) and Monteiro and Page (2008) among others. A number of papers appeared in the topic of discontinuous games and further extensions have been obtained in several directions; see, for example, Lebrun (1996), Bagh and Jofre (2006), Bich (2009), Carbonell-Nicolau (2011), Balder (2011), Carmona (2010, 2011), Carmona (forthcoming), Prokopovych (2011, 2013), Prokopovych (forthcoming), de Castro (2011), McLennan et al. (2011), Reny (2011, 2013), Tian (2012), Barelli and Meneghel (2013) and Prokopovych and Yannelis (2014) among others.

In this paper, we consider discontinuous games with asymmetric information; i.e., games with a finite set of players and each of whom is characterized by his own private information (which is a partition of an exogenously given state space representing the uncertainty of the world), a strategy set, a state dependent (random) utility function and a prior. This problem arises naturally in situations where privately informed agents behave strategically. Because of its importance, the research trend in this field has been quite active since Harsanyi's seminal work. The main purpose of this paper is to provide new equilibrium existence result for Bayesian games with discontinuous payoffs.

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We introduce the notions of finite/finite* payoff security and adopt the aggregate upper semicontinuity condition in the ex post games. We show that the (ex ante) Bayesian game is payoff secure and reciprocal upper semicontinuous, and hence Reny's theorem is applicable and a pure strategy Bayesian equilibrium exists. A key issue here is that the quasiconcavity of the Bayesian game cannot be guaranteed even if all ex post games are quasiconcave. We show by means of counterexamples that the concavity and finite payoff security conditions of the ex post games are both necessary for the existence of a pure strategy Bayesian equilibrium.[1]

The rest of the paper is organized as follows. In Section 2, we introduce a discontinuous game with asymmetric information and relevant results in deterministic discontinuous games. In Section 3, we prove the existence of a Bayesian equilibrium. Section 4 collects the discussions on the conditions of Theorem 2, the comparison of our notion with the uniform payoff security condition of Monteiro and Page (2007), and possible extensions to the case of a continuum of states. Some concluding remarks are collected in Section 5.

2. Model

2.1. Discontinuous games with asymmetric information

We consider an **asymmetric information game**

$$G = \{\Omega, (u_i, X_i, \mathcal{F}_i)_{i\in I}\}.$$

- There is a finite set of players, $I = \{1, 2, \cdots, N\}$.
- Ω is a countable state space representing the **uncertainty** of the world, $\mathcal{F}$ is the power set of Ω.
- $\mathcal{F}_i$ is a partition of Ω, denoting the **private information** of player i. $\mathcal{F}_i(\omega)$ denotes the element of $\mathcal{F}_i$ including the state ω.
- Player i's action space X_i is a nonempty, compact, convex subset of a topological vector space, $X = \prod_{i\in I} X_i$.
- For every $i \in I$, $u_i : X \times \Omega \to \mathbb{R}$ is a **random utility function** representing the (ex post) preference of player i.

A game G is called a **compact** game if u_i is bounded for every $i \in I$; i.e., $\exists M > 0$, $|u_i(x, \omega)| \le M$ for all $x \in X$ and $\omega \in \Omega$, $1 \le i \le N$. A game G is said to be **quasiconcave (resp. concave)** if $u_i(\cdot, x_{-i}, \omega)$ is quasiconcave (resp. concave) for every $i \in I$, $x_{-i} \in X_{-i}$ and $\omega \in \Omega$. For every $\omega \in \Omega$, the **ex post game** is $G_\omega = (u_i(\cdot, \omega), X_i)_{i\in I}$. Suppose that each player has a **private prior** π_i on $\mathcal{F}$ such that $\pi_i(E) > 0$ for any $E \in \mathcal{F}_i$. The **weighted ex post game** is $G'_\omega = (w_i(\cdot, \omega), X_i)_{i\in I}$, where $w_i(\cdot, \omega)$ is a mapping from X to $\mathbb{R}$ and $w_i(\cdot, \omega) = u_i(\cdot, \omega)\pi_i(\omega)$ for each $\omega \in \Omega$.

For every player i, a **strategy** is an $\mathcal{F}_i$-measurable mapping from Ω to X_i. Let

$$L_i = \{f_i : \Omega \to X_i : f_i \text{ is } \mathcal{F}_i\text{-measurable}\},$$

then L_i is a convex and compact set endowed with the product topology. $L = \prod_{i\in I} L_i$. Given a strategy profile $f \in L$, the **expected utility** of player i is

$$U_i(f) = \sum_{\omega\in\Omega} u_i(f_i(\omega), f_{-i}(\omega), \omega)\pi_i(\omega),$$

then $U_i(\cdot)$ is also bounded by M. Therefore, the **(ex ante) Bayesian game** of G is $G_0 = (U_i, L_i)_{1\le i\le N}$, which is compact and concave if the game G is compact and concave. A strategy profile $f \in L$ is said to be a **Bayesian equilibrium** if for each player i and any $g_i \in L_i$,

$$U_i(f) \ge U_i(g_i, f_{-i}).$$

Remark 1. It is well known that quasiconcavity may not be preserved under summation or integration. Thus, the Bayesian game G_0 may not be quasiconcave even if G is quasiconcave.

2.2. Deterministic case

Hereafter, $G_d = (X_i, u_i)_{i=1}^N$ will denote a **deterministic discontinuous game**, i.e., Ω is a singleton. The following definitions strengthen the notion of payoff security in Reny (1999).

Definition 1. In the game G_d, player i can secure an n-dimensional payoff $(\alpha_1, \cdots, \alpha_n) \in \mathbb{R}^n$ at $(x_i, x^1_{-i}, \cdots, x^n_{-i}) \in X_i \times X^n_{-i}$ if there is $\overline{x_i} \in X_i$, such that $u_i(\overline{x_i}, y^k_{-i}) \ge \alpha_k$ for all y^k_{-i} in some open neighborhood of x^k_{-i}, $1 \le k \le n$.

[1] Based on a different approach using the communication device, Jackson et al. (2002) also studied discontinuous games with asymmetric information.

Definition 2. The game G_d is ***n*-payoff secure** if for every $i \in I$ and $(x_i, x_{-i}^1, \cdots, x_{-i}^n) \in X_i \times X_{-i}^n$, $\forall \epsilon > 0$, player i can secure an n-dimensional payoff

$$\left(u_i(x_i, x_{-i}^1) - \epsilon, \cdots, u_i(x_i, x_{-i}^n) - \epsilon\right)$$

at $(x_i, x_{-i}^1, \cdots, x_{-i}^n) \in X_i \times X_{-i}^n$. The game G_d is said to be **finitely payoff secure** if it is n-payoff secure for any $n \in \mathbb{N}$.[2]

If $n = 1$, it is called **payoff secure**.

Given $x \in X$, let $u(x) = (u_1(x), \cdots, u_N(x))$ be the payoff vector of the game G_d. Define $\Gamma_d = \{(x, u(x)) \in X \times \mathbb{R} : x \in X\}$, i.e., the graph of the payoff vector $u(\cdot)$, then $\overline{\Gamma_d}$ denotes the closure of Γ_d.

The following definition is due to Reny (1999).

Definition 3. The game G_d is **better-reply secure** if whenever $(x^*, \alpha^*) \in \overline{\Gamma_d}$ and x^* is not a Nash equilibrium, some player j can secure a payoff strictly above α_i^* at x^*.

In their pioneer paper, Dasgupta and Maskin (1986) proposed the following condition which is weaker than the upper semicontinuity condition of the utility functions.

Definition 4. A game G_d is said to be **aggregate upper semicontinuous** if the summation of the utility functions of all players is upper semicontinuous.[3]

The following generalization is due to Simon (1987), which is called complimentary discontinuity or reciprocal upper semicontinuity.

Definition 5. A game G_d is **reciprocal upper semicontinuous** if for any $(x, \alpha) \in \overline{\Gamma_d} \setminus \Gamma_d$, there is a player i such that $u_i(x) > \alpha_i$.

Reny (1999) showed that the game G_d is better reply secure if it is payoff secure and reciprocal upper semicontinuous.

Theorem 1. *(See Reny (1999).) Every compact, quasiconcave and better-reply secure deterministic game has a Nash equilibrium.*

We will use this theorem to establish our existence results. One may easily develop analogous definitions of "n-payoff security" in the framework of many recent papers.

3. Existence of Bayesian equilibrium

In this section, we will show the existence of pure strategy Bayesian equilibrium in discontinuous games with asymmetric information.

First, we shall prove Propositions 1 and 2, which provide sufficient conditions to guarantee the payoff security of a Bayesian game.

Proposition 1. *If the weighted ex post game G'_ω is finitely payoff secure at every state $\omega \in \Omega$ and $u_i(x, \cdot)$ is $\mathcal{F}_i$-measurable for every $x \in X$ and $i \in I$, then the Bayesian game G_0 is payoff secure.*

Proof. For any $i \in I$, suppose that $\mathcal{F}_i = \{E_1, \cdots, E_k, \cdots\}$ is the information partition of player i, M is the bound for u_i. Given any $\epsilon > 0$, there exists a positive integer $K > 0$ such that $\pi_i(\cup_{k=1}^K E_k) > 1 - \frac{\epsilon}{6M}$. For $1 \le k \le K$, there exists a finite subset $E'_k \subseteq E_k$ such that $\pi_i(E_k \setminus E'_k) < \frac{\epsilon}{6KM}$ and $\pi_i(E'_k) > 0$.

Fix $\omega_k \in E'_k$ such that $\pi_i(\omega_k) > 0$. Given any $f \in L$, because $u_i(x, \cdot)$ and $f_i(\cdot)$ are both $\mathcal{F}_i$-measurable,

$$u_i(f_i(\omega), f_{-i}(\omega), \omega) = u_i(f_i(\omega_k), f_{-i}(\omega), \omega_k)$$

for any $\omega \in E_k$, $1 \le k \le K$.

Since G'_{ω_k} is finitely payoff secure, there exists a point $y_i^k \in X_i$, such that

$$w_i(y_i^k, y_{-i}^\omega, \omega_k) \ge w_i(f_i(\omega_k), f_{-i}(\omega), \omega_k) - \frac{\epsilon}{3}\pi_i(\omega_k)$$

for all y_{-i}^ω in some open neighborhood O_ω of $f_{-i}(\omega)$, $\forall \omega \in E'_k$.

[2] It is clear that the uniform payoff security condition of Monteiro and Page (2007) implies our finite payoff security condition. See Section 4.2 for further discussion of this point.

[3] Carmona (2009) proved the existence of Nash equilibria in compact, quasiconcave games via weak versions of both upper semicontinuity and payoff security.

Let

$$g_i(\omega)=\begin{cases} y_i^k, & \text{if } \omega\in E_k \text{ for } 1\le k\le K,\\ f_i(\omega), & \text{otherwise.}\end{cases}$$

Then by construction g_i is $\mathcal{F}_i$-measurable.

Choose the open set O in L_{-i} such that $O=\left(\prod_{1\le k\le K}(\prod_{\omega\in E'_k} O_\omega\times X_{-i}^{E_k\setminus E'_k})\right)\times X_{-i}^{\Omega\setminus\cup_{1\le k\le K}E_k}$,

$$\begin{aligned}
U_i(g_i,g'_{-i}) &= \sum_{E\in\mathcal{F}_i}\sum_{\omega\in E} w_i(g_i(\omega),g'_{-i}(\omega),\omega)\\
&\ge \sum_{k=1}^{K}\sum_{\omega\in E'_k} w_i(g_i(\omega),g'_{-i}(\omega),\omega)-M\left(\pi_i(\Omega\setminus(\cup_{k=1}^K E_k))+\sum_{k=1}^{K}\pi_i(E_k\setminus E'_k)\right)\\
&\ge \sum_{k=1}^{K}\sum_{\omega\in E'_k} w_i(y_i^k,g'_{-i}(\omega),\omega_k)\frac{\pi_i(\omega)}{\pi_i(\omega_k)}-\frac{\epsilon}{3}\\
&\ge \sum_{k=1}^{K}\sum_{\omega\in E'_k} [w_i(f_i(\omega_k),f_{-i}(\omega),\omega_k)-\frac{\epsilon}{3}\pi_i(\omega_k)]\frac{\pi_i(\omega)}{\pi_i(\omega_k)}-\frac{\epsilon}{3}\\
&\ge \sum_{k=1}^{K}\sum_{\omega\in E'_k} w_i(f_i(\omega_k),f_{-i}(\omega),\omega)-\frac{2\epsilon}{3}\\
&\ge \sum_{E\in\mathcal{F}_i}\sum_{\omega\in E} w_i(f_i(\omega_k),f_{-i}(\omega),\omega)-\frac{2\epsilon}{3}-M\left(\pi_i(\Omega\setminus(\cup_{k=1}^K E_k))+\sum_{k=1}^{K}\pi_i(E_k\setminus E'_k)\right)\\
&> U_i(f)-\epsilon
\end{aligned}$$

for every $g'_{-i}\in O$. Thus, the game G_0 is payoff secure. □

Remark 2. Note that the finitely payoff security of the weighted ex post game $G'_\omega=(w_i(\cdot,\omega),X_i)_{i\in I}$ is slightly weaker than the finitely payoff security of the ex post game $G_\omega=(u_i(\cdot,\omega),X_i)_{i\in I}$, where u_i is the ex post payoff function and $w_i(\cdot,\omega)=u_i(\cdot,\omega)\cdot\pi_i(\omega)$ for every $i\in I$. These two conditions will be equivalent if $\pi_i(\omega)>0$ for any $i\in I$ and $\omega\in\Omega$.

In Proposition 1, the ex post utility functions are required to be private information measurable. This assumption can be dropped if the finitely payoff security condition is strengthened accordingly.

Definition 6. An asymmetric information game G is n^*-**payoff secure** if for every $i\in I$, every $(x_i,x^1_{-i},\cdots,x^n_{-i})\in X_i\times X^n_{-i}$ and every $(\omega_1,\cdots,\omega_n)\subseteq D$ for some $D\in\mathcal{F}_i$, $\forall\epsilon>0$, there is $\overline{x_i}\in X_i$, such that $u_i(\overline{x_i},y^k_{-i},\omega_k)\ge u_i(x_i,x^k_{-i},\omega_k)-\epsilon$ for all y^k_{-i} in some open neighborhood of x^k_{-i}, $1\le k\le n$.

The game G is said to be **finitely*** **payoff secure** if it is n^*-payoff secure for any $n\in\mathbb{N}$.

Proposition 2. *The Bayesian game G_0 is payoff secure if G is finitely*-payoff secure.*

Proof. As in the proof of Proposition 1, we could find some positive integer K and finite set E'_k for each $1\le k\le K$ satisfying the same conditions therein.

Given any $f\in L$. Since G is finitely* payoff secure, for each $1\le k\le K$, there exists a point $y_i^k\in X_i$, such that

$$u_i(y_i^k,y^\omega_{-i},\omega)\ge u_i(f_i(\omega),f_{-i}(\omega),\omega)-\frac{\epsilon}{3}$$

for all y^ω_{-i} in some open neighborhood O_ω of $f_{-i}(\omega)$, $\forall\omega\in E'_k$.

Let

$$g_i(\omega)=\begin{cases} y_i^k, & \text{if } \omega\in E_k \text{ for } 1\le k\le K,\\ f_i(\omega), & \text{otherwise.}\end{cases}$$

Then the rest of the proof proceeds similarly as in the proof of Proposition 1. □

Proposition 3. *In the game G, if the weighted ex post game G'_ω is aggregate upper semicontinuous at every state $\omega \in \Omega$, then the Bayesian game G_0 is reciprocal upper semicontinuous.*

Proof. By way of contradiction, suppose that the Bayesian game G_0 is not reciprocal upper semicontinuous. Then there exists a sequence $\{f^n\} \subseteq L$, $f^n \to f$ and $U(f^n) \to \alpha$ as $n \to \infty$, where $U(f) = (U_1(f), \cdots, U_N(f))$ and $\alpha = (\alpha_1, \cdots, \alpha_N) \in \mathbb{R}^N$. $U_i(f) \leq \alpha_i$ for $1 \leq i \leq N$ and $U(f) \neq \alpha$.

Denote $\epsilon = \max_{1 \leq i \leq N}(\alpha_i - U_i(f))$, $\epsilon > 0$. Thus,

$$\sum_{i \in I} U_i(f) \leq \sum_{i \in I} \alpha_i - \epsilon.$$

There exists a finite subset $E \subseteq \Omega$ such that $\pi_i(\Omega \setminus E) < \frac{\epsilon}{2NM}$ for every $i \in I$, where M is the bound of u_i for all i.

Then for any $i \in I$, $U_i(f^n)$ can be divided into two parts: $\mu_i^n = \sum_{\omega \in E} w_i(f^n(\omega), \omega)$ and $\nu_i^n = \sum_{\omega \notin E} w_i(f^n(\omega), \omega)$, $U_i(f^n) = \mu_i^n + \nu_i^n$. Let $\mu^n = \{\mu_i^n\}_{i \in I}$, since $\{\mu^n\}_{n \in \mathbb{N}}$ is bounded, there is a subsequence, say itself, which converges to some $\mu \in \mathbb{R}^N$. Since $\nu_i^n \leq M\pi_i(\Omega \setminus E) < \frac{\epsilon}{2N}$ for any $i \in I$ and $n \in \mathbb{N}$, $\mu_i \geq \alpha_i - \frac{\epsilon}{2N}$ for every $i \in I$.

At each state $\omega \in E$ and $i \in I$, since $w_i(f^n(\omega), \omega)$ is bounded, there is a subsequence which converges to some β_i^ω. Since there are only finitely many players and states, we can assume without loss of generality that $w_i(f^n(\omega), \omega) \to \beta_i^\omega$ as $n \to \infty$, then $\sum_{\omega \in E} \beta_i^\omega = \mu_i$.

Since $f^n(\omega) \to f(\omega)$ for every $\omega \in E$ and G'_ω is aggregate upper semicontinuous,

$$\sum_{i \in I} w_i(f(\omega), \omega) \geq \sum_{i \in I} \beta_i^\omega.$$

Thus,

$$\sum_{i \in I} U_i(f) \geq \sum_{i \in I} \sum_{\omega \in E} w_i(f(\omega), \omega) \geq \sum_{i \in I} \sum_{\omega \in E} \beta_i^\omega = \sum_{i \in I} \mu_i \geq \sum_{i \in I} \alpha_i - \frac{\epsilon}{2},$$

which is a contradiction. □

By combining Theorem 1, Propositions 1, 2 and 3, we obtain the following result which is an extension of Reny (1999) to Bayesian games with discontinuous payoffs.

Theorem 2. *Suppose that an asymmetric information game G is compact, the corresponding Bayesian game G_0 is quasiconcave, and the weighted ex post game G'_ω is aggregate upper semicontinuous at each state ω. Then a Bayesian equilibrium exists if either of the following conditions holds.*

1. *The weighted ex post game G'_ω is finitely payoff secure at every state $\omega \in \Omega$ and $u_i(x, \cdot)$ is $\mathcal{F}_i$-measurable for every $x \in X$ and $i \in I$.*
2. *The game G is finitely* payoff secure.*

Remark 3. Note that the (ex ante) Bayesian game G_0 is assumed to be quasiconcave. However, Example 1 below indicates that the theorem may fail if we only require that G is quasiconcave. To impose conditions in the primitive stage, one possible alternative is to require that G be concave. However, the concavity of the utility function implies that it is continuous on the interior of its domain, and hence the discontinuity only arises on the boundary. This is a rather strong assumption and will deter many possible applications.

4. Discussion

In this section, we shall first provide two examples to show the necessity of the quasiconcavity and the finite payoff security conditions. In addition, we shall also compare our notion of finite payoff security and the uniform payoff security condition of Monteiro and Page (2007), and discuss the possible extension of Theorem 2 to the setting of a continuum of states based on the uniform payoff security condition.

4.1. Two counterexamples

To guarantee the existence of a Bayesian equilibrium, the expected utility of each player is required to be quasiconcave in Theorem 2. Example 1 below shows that this condition cannot be dropped, even if all other conditions are satisfied and the ex post utility function is quasiconcave itself.

Example 1 *(Importance of concavity).* Consider the following game G. There are two players $I = \{1, 2\}$ competing for an object. The strategy spaces for players 1 and 2 are respectively X and Y, $X = Y = [0, 1]$. Player 1 has only one possible private value 1, and player 2 has two possible private values 0 and 1.

Denote $a = (1, 1)$ and $b = (1, 0)$ (the first component is the private value of player 1 and the second component is the private value of player 2). The state space is $\Omega = \{a, b\}$. The information partitions and priors are as follows:

$$\mathcal{F}_1 = \{\{a, b\}\}, \pi_1(a) = \pi_1(b) = \frac{1}{2};$$

$$\mathcal{F}_2 = \{\{a\}, \{b\}\}, \pi_2(a) = \pi_2(b) = \frac{1}{2}.$$

For $\omega = a, b$, the utility function of player 1 is

$$u_1(x, y, \omega) = \begin{cases} 1 - x, & \text{if } x \geq y \\ 0, & \text{otherwise.} \end{cases}$$

Then $u_1(x, y, \cdot)$ is measurable with respect to $\mathcal{F}_1$ for any $(x, y) \in X \times Y$.

The utility function of player 2 is

$$u_2(x, y, a) = \begin{cases} 1 - y, & \text{if } y > x \\ 0, & \text{if } y \leq x \end{cases}$$

and

$$u_2(x, y, b) = \begin{cases} -y, & \text{if } y > x \\ \frac{-y}{2}, & \text{if } y \leq x. \end{cases}$$

1. At both states, when there is a tie, player 1 will take the good and player 2 gets nothing.
2. At state b, the private value of player 2 is 0, bidding for positive price will harm both, thus player 2 will be punished when he bids more than 0 even if he loses the game.

The ex post games G_a and G_b are 2-payoff secure. Consider the ex post game G_a and player 1. Given $\epsilon > 0$, $x \in X$ and $(y_1, y_2) \in Y \times Y$. Assume $y_1 \geq y_2$ without loss of generality. There are three possible cases.

1. If $y_1 \leq x$, then let player 1 bid $\bar{x} = \min\{x + \frac{\epsilon}{2}, 1\}$. For $i = 1, 2$, $y_i' \leq \min\{x + \frac{\epsilon}{2}, 1\}$ for any y_i' in a small neighborhood of y_i, hence the payoff of player 1 is at least $1 - x - \frac{\epsilon}{2}$.
2. If $y_2 > x$, then let player 1 bid $\bar{x} = x$ and his payoff cannot be worse off.
3. If $y_2 \leq x < y_1$, then let player 1 bid $\bar{x} = x + \delta$ such that $x + \delta < y_1$ and $0 < \delta < \epsilon$.

Similarly, one can show the 2-payoff security of player 2 at state a and b. Therefore, the ex post game is 2-payoff secure at each state. It is easy to see that the summations of ex post utility functions are upper semicontinuous at both states, and the assumptions of quasiconcavity and compactness are satisfied. Thus, there are Nash equilibria for both ex post games. At state a, the unique equilibrium is $(1, 1)$; at state b, the unique equilibrium is $(0, 0)$.

However, there is no Bayesian equilibrium in this game.[4] Suppose (x, y) is an equilibrium, where $y = (y(a), y(b))$. In state b, player 2 will always choose $y(b) = 0$, thus player 1 can guarantee himself a positive payoff by choosing $x = 0$. But if $x < 1$, player 2 has no optimal strategy at state a. Thus, player 1 has to choose $x = 1$ and gets 0 payoff, a contradiction.

Remark 4. In Example 1, although the ex post utility function is quasiconcave at both states, the expected utility function is not quasiconcave, and hence there is no Bayesian equilibrium.

In Theorem 2, we strengthen the payoff security of Reny (1999) to finite payoff security. The second example shows that the payoff security of every ex post game cannot guarantee the payoff security of the Bayesian game.

Example 2 *(Ex post payoff security does not imply ex ante payoff security).* Consider the following game: the player space is $I = \{1, 2, 3\}$, the state space is $\Omega = \{a, b\}$, and the information partitions of all players are $\mathcal{F}_1 = \mathcal{F}_2 = \{\{a, b\}\}$ and $\mathcal{F}_3 = \{\{a\}, \{b\}\}$. Players have common prior $\pi(a) = \pi(b) = \frac{1}{2}$. The action space of player i is $X_i = [0, 1]$, $i = 1, 2, 3$. The games L and R are listed below.

In both states, players 1 and 2 will play the game L if $x_3 = 0$ and the game R otherwise. Player 1's action is in the left and player 2's action is in the top.

[4] Note that there is mixed strategy equilibria for this game: for example, Bidder 1's strategy is $\frac{1}{2}\delta_0 + \frac{1}{2}U([0, \frac{1}{2}])$, Bidder 2's strategy is 0 when his value is 0, and $U([0, \frac{1}{2}])$ when his value is 1, where δ_0 is the delta measure concentrated at 0 and $U([0, \frac{1}{2}])$ is the uniform distribution on $[0, \frac{1}{2}]$. However, we only focus on pure strategies in this paper.

L:

	0	(0,1)	1
0	(1, 6)	(0, 7)	(3, 7)
(0, 1)	(5, 4)	(4, 5)	(3, 7)
1	(6, 3)	(6, 3)	(4, 5)

R:

	0	(0,1)	1
0	(14, 14)	(16, 10)	(16, 10)
(0, 1)	(13, 15)	(14, 14)	(14, 14)
1	(13, 15)	(12, 15)	(12, 15)

The utility function of player 3 is defined as follow:

$$u_3(x_1, x_2, x_3, \omega) = \begin{cases} 1, & \text{if } x_3 = 0 \text{ at } \omega = a \text{ or } x_3 \in (0,1] \text{ at } \omega = b; \\ 0, & \text{otherwise.} \end{cases}$$

Below we study the ex post game G_a and show that it is payoff secure but not 2-payoff secure. The same result holds for the ex post game G_b. However, the Bayesian game is not payoff secure.

In the game L, player 1 can choose the dominant strategy $x_1 = 1$ and player 2 can choose the dominant strategy $x_2 = 1$, thus the game L is payoff secure. In the game R, player 1 can choose the dominant strategy $x_1 = 0$ and player 2 can choose the dominant strategy $x_2 = 0$, thus the game R is payoff secure.

Suppose state a realizes. The payoff of player 3 is secured since he can always choose $x_3 = 0$, which could guarantee his highest payoff. For players 1 and 2, if player 3's action $x_3 = 0$, then players 1 and 2 will play the game L and it is payoff secure since if x_3 deviates in a small neighborhood, then players 1 and 2 will play the game R and their payoffs are strictly higher; if x_3 stays unchanged and they are still in game L, then the payoff security of the game L supports our claim. If player 3's action $x_3 \in (0, 1]$, they will play game R and it is payoff secure since a sufficiently small neighborhood of x_3 is still included in $(0, 1]$ and the game R itself is payoff secure. Therefore, the ex post game G_a is payoff secure.

However, this game is not 2-payoff secure. For example, let $x_1 = 0$, $(x_2^1, x_3^1) = (1, 0)$ and $(x_2^2, x_3^2) = (1, 1)$, there is no action which could guarantee that player 1 can secure the 2 dimensional payoff vector $(3, 16)$. Similarly, one could show that the ex post game G_b is also payoff secure but not 2-payoff secure.

Finally, we verify our claim that the Bayesian game is not payoff secure. Let the strategy of player 3 be $x_3 = (x_3(a), x_3(b)) = (0, 1)$, the expected utilities for players 1 and 2 are listed as the following game E.

	0	(0,1)	1
0	$(\frac{15}{2}, 10)$	$(8, \frac{17}{2})$	$(\frac{19}{2}, \frac{17}{2})$
(0, 1)	$(9, \frac{19}{2})$	$(9, \frac{19}{2})$	$(\frac{17}{2}, \frac{21}{2})$
1	$(\frac{19}{2}, 9)$	$(9, 9)$	$(8, 10)$

E

Then player 1 cannot secure his payoff if $x_1 = 1$ and $x_2 = 0$, and player 2 cannot secure his payoff if $x_1 = 0$ and $x_2 = 0$.

Moreover, this game does not have a Bayesian equilibrium. It is easy to see that player 3 will choose $x_3(a) = 0$ and $x_3(b) \in (0, 1]$. Consequently, the expected payoff matrix of players 1 and 2 is E. However, the game E has no equilibrium.

Remark 5. The game in Example 2 is obviously compact and satisfies the private information measurability requirement. We need to show that the Bayesian game is quasiconcave. It is clear that the expected utility of player 3 is quasiconcave. Now we consider players 1 and 2. Given $x_3 = (x_3(a), x_3(b))$. If $x_3 = (0, 0)$, then players 1 and 2 will play the game L in both states. Their expected payoff matrix is L, which is quasiconcave. If $x_3 \in (0, 1] \times (0, 1]$, players 1 and 2 will play the game R in both states, and hence their expected payoff matrix is the quasiconcave game R. Otherwise, players 1 and 2 will play the game L at one state and the game R at the other state. That is, their expected payoff matrix is E, which is also quasiconcave.

4.2. Comparison with Monteiro and Page (2007)

Below, we compare our notion of finite payoff security with the uniform payoff security of Monteiro and Page (2007). The following condition is due to Monteiro and Page (2007).

Definition 7. The game G_d is **uniform payoff secure** if for every $i \in I$ and $x_i \in X_i$, $\forall \epsilon > 0$, there is $\overline{x_i} \in X_i$ such that for every $x_{-i} \in X_{-i}$, $u_i(\overline{x_i}, y_{-i}) \geq u_i(x_i, x_{-i}) - \epsilon$ for all y_{-i} in some open neighborhood of x_{-i}.

A game G is uniformly payoff secure if each player starting at any strategy $x_i \in X_i$ has a strategy $\overline{x_i} \in X_i$ he can move to in order to secure a payoff of $u_i(x_i, x_{-i})$ against all possible small deviations of all strategy profiles of others. It is obvious that the uniform payoff security condition is stronger than our finite payoff security condition. Below, we provide an example which shows that the uniform payoff security is strictly stronger than the finite payoff security condition.

Example 3. Given a deterministic game G such that $I = \{1, 2\}$, $X_1 = X_2 = [0, 1]$,

$$u_1(x_1, x_2) = \begin{cases} -1, & \text{if } x_1 < x_2 < \frac{1}{2}(x_1 + 1); \\ 0, & \text{if } x_1 = x_2 \text{ or } x_1 = 2x_2 - 1; \\ 1, & \text{otherwise} \end{cases}$$

and $u_2 \equiv 0$.

We shall show that this game is finitely payoff secure, but not uniformly payoff secure. We only need to verify this for player 1. Fix arbitrary $n \in \mathbb{N}$. Pick $(x_1, x_2^1, \ldots, x_2^n) \in X_1 \times X_2^n$. Without loss of generality, assume that $x_2^1 < x_2^2 < \cdots < x_2^n$. If $x_2^n < 1$, then choose $\overline{x_1} = 1$; if $x_2^n = 1$, then choose $\overline{x_1}$ sufficiently close to 1 such that $x_2^{n-1} < \overline{x_1} < 1$. In all these cases we can find a neighborhood $O_{x_2^k}$ of x_2^k such that $u_1(\overline{x_1}, y_2^k) \geq u_1(x_1, x_2^k)$ for all $y_2^k \in O_{x_2^k}$, $1 \leq k \leq n$.

However, the uniform payoff security condition is not satisfied in this game. Thus, the uniform payoff security condition is strictly stronger than the finite payoff security condition.

4.3. Extension of Theorem 2 to a continuum of states

By modifying the uniform payoff security condition of Monteiro and Page (2007) and adopting the standard absolute continuity condition of Milgrom and Weber (1985), Carbonell-Nicolau and McLean (2014) proved the existence of behavioral/distributional strategy Bayesian equilibrium in the setting of a continuum of states. They do not need to impose the quasiconcavity condition on the payoffs since the concavity property is automatic by working with behavioral/distributional strategies. We will show that our Theorem 2 can be extended to the setting of a continuum of states by strengthening the finite payoff security to uniform payoff security.

The model of **Bayesian games with a continuum of states** is as follows.

- The set of players: $I = \{1, 2, \ldots, N\}$.
- The set of **actions** available to each player: $\{X_i\}_{i \in I}$. Each X_i is a nonempty compact metric space endowed with the Borel σ-algebra $\mathcal{B}(X_i)$. Let $X = \times_{i=1}^n X_i$ and $\mathcal{B}(X) = \otimes_{i \in I} \mathcal{B}(X_i)$.
- The **(private) information space** for each player: T_i. Each T_i is a measurable space endowed with a σ-algebra $\mathcal{T}_i$. Let $T = \times_{i=1}^n T_i$ and $\mathcal{T} = \otimes_{i=1}^n \mathcal{T}_i$.
- The **payoff functions**: $\{u_i\}_{i \in I}$. Each $u_i \colon X \times T \to \mathbb{R}$ is a bounded measurable mapping.
- The **information structure**: λ, a probability measure on the measurable space $(T, \mathcal{T})$ with marginal λ_i on T_i for each $i \in I$.

The following condition is an extension of Definition 7 to the case of incomplete information games, and it is due to Carbonell-Nicolau and McLean (2014). Based on this condition, Carbonell-Nicolau and McLean (2014) proved the existence of a behavioral strategy equilibrium (see Theorem 1 therein).

Definition 8. The Bayesian game is uniformly payoff secure if for each $i \in I$, $\epsilon > 0$, and a behavioral strategy f_i, there exists another behavioral strategy g_i such that for all (t, x_{-i}), there exists a neighborhood $V_{x_{-i}}$ of x_{-i} such that

$$u_i(t, g_i(t_i), y_{-i}) > u_i(t, f_i(t_i), x_{-i}) - \epsilon$$

for all $y_{-i} \in V_{x_{-i}}$.

Below, we consider the purification of behavioral strategies. He and Sun (2014) proposed the "relative diffuseness" condition as a characterization of the relation between two kinds of diffuseness of information, and proved a purification theorem for Bayesian games based on this condition.

For each $i \in I$, let $(T_i, \mathcal{T}_i, \lambda_i)$ be the private information space, and $\mathcal{F}_i \subseteq \mathcal{T}_i$ be the smallest σ-algebra relative to which u_i is measurable. The σ-algebras $\mathcal{T}_i$ and $\mathcal{F}_i$ will represent the diffuseness of information from the aspect of strategies and from the aspect of payoffs, respectively. The probability spaces $(T_i, \mathcal{T}_i, \lambda_i)$ and $(T_i, \mathcal{F}_i, \lambda_i)$ will be used to model the information space and the payoff-relevant information space.

For any nonnegligible subset $D \in \mathcal{T}_i$, the restricted probability space $(D, \mathcal{F}_i^D, \lambda_i^D)$ is defined as follows: $\mathcal{F}_i^D$ is the σ-algebra $\{D \cap D' : D' \in \mathcal{F}_i\}$ and λ_i^D the probability measure re-scaled from the restriction of λ_i to $\mathcal{F}_i^D$. Furthermore, $(D, \mathcal{T}_i^D, \lambda_i^D)$ can be defined similarly.

Definition 9. Following the notations above, $\mathcal{F}_i$ is said to be **setwise coarser** than $\mathcal{T}_i$ if for every $D \in \mathcal{T}_i$ with $\lambda_i(D) > 0$, there exists a $\mathcal{T}_i$-measurable subset D_0 of D such that $\lambda_i(D_0 \triangle D_1) > 0$ for any $D_1 \in \mathcal{F}_i^D$.

The following assumption due to He and Sun (2014) states that on any nonnegligible set $D \subseteq T_i$, $\mathcal{T}_i^D$ is always larger than $\mathcal{F}_i^D$. That is, the strategy-relevant diffuseness of information is essentially richer than the payoff-relevant diffuseness of information.

Assumption (RD). For each $i \in I$, $(T_i, \mathcal{T}_i, \lambda_i)$ is atomless and $\mathcal{F}_i$ is setwise coarser than $\mathcal{T}_i$.

Now we are ready to prove the existence of a pure strategy Bayesian equilibrium with a continuum of states.

Theorem 3. *Suppose that*

1. *Assumption (RD) holds,*[5] *u_i is measurable with respect to $\mathcal{F}_i$ for each $i \in I$, and $\lambda = \otimes_{i \in I} \lambda_i$;*
2. *the Bayesian game is uniformly payoff secure and that each $t \in T$, the map $\sum_{i \in I} u_i(t, \cdot) : X \to \mathbb{R}$ is upper semicontinuous.*

Then there exists a pure strategy Bayesian equilibrium.

Proof. By Theorem 1 of Carbonell-Nicolau and McLean (2014), there exists a behavioral strategy Bayesian equilibrium f, and by Theorem 2 of He and Sun (2014), f has a purification g, which is a pure strategy Bayesian equilibrium. □

Remark 6. In an incomplete information game with finitely many states, one can work with the k-payoff security condition, where k could be the number of all states in the Bayesian game. However, as we consider a Bayesian game with countably many states, we need to extend the k-payoff security to finite payoff security as we may need to use a Bayesian game with arbitrarily finitely many states to approximate the original Bayesian game. Monteiro and Page (2007) proved the existence of a mixed strategy equilibrium $m = (m_1, \ldots, m_n)$ with the stronger condition of uniform payoff security in a simple deterministic setting. Indeed, their result can be understood as an existence result of a pure strategy Bayesian equilibrium in a Bayesian game with uncountable states and state-irrelevant payoffs. Thus, they need to further strengthen the condition due to the larger size of the state space.

In particular, suppose that each player can only observe his own private signal from the unit interval $[0, 1]$, which is endowed with the uniform distribution η. Let $\Omega = [0, 1]^n$ be the state space. The payoff of each player only depends on the action profile, but not on the state. Then the deterministic game is reformulated as a Bayesian game with uncountable states and state-irrelevant payoffs.

The mixed strategy m_i of player i in the deterministic game can be realized by his private signal (like a randomization device) to be a pure strategy f_i in the sense that $m_i = \eta \circ f_i^{-1}$. It is easy to check that $f = (f_1, \ldots, f_n)$ is a pure strategy Bayesian equilibrium in this Bayesian game.

If we view a deterministic discontinuous game as such a Bayesian game, then $\mathcal{F}_i = \{\emptyset, [0, 1]\}$ for each $i \in I$ since players' payoffs do not depend on the states. Thus, our Assumption (RD) holds trivially, and our result goes beyond Monteiro and Page (2007) by allowing for the payoffs to be state-dependent.[6]

Remark 7. If one views a deterministic game as a Bayesian game with uncountable states and let f_i and g_i be pure strategies and state irrelevant in Definition 8, then this condition reduces to the uniform payoff security in the sense of Monteiro and Page (2007). In Theorem 3, we adopt the uniform payoff security condition in the sense of Carbonell-Nicolau and McLean (2014) since our payoffs are state dependent, and thus the best response of each player is typically state dependent. Therefore, one needs to compare the state dependent strategies for each player; for more discussions, see Carbonell-Nicolau and McLean (2014).

5. Concluding remarks

Our purpose was to impose the same assumptions of Reny (1999) on primitives and prove new equilibrium existence theorems for Bayesian games. We showed that if players are Bayesians, the conditions of Reny (1999) need to be strengthened. By introducing a new payoff security condition which is a strengthening of the one of Reny (1999), we showed that if our new condition is imposed on the weighted ex post utility functions, then the ex ante expected utility is payoff secure. In view of this result and by assuming that the expected utilities are quasiconcave, we proved an equilibrium existence theorem for discontinuous games with asymmetric information. Also, we pointed out that the concavity assumption plays an important role; specifically, if the ex post utility function of each player is not concave, then a Bayesian equilibrium may not exist

The results of the current paper can be extended to a social system or abstract economies à la Debreu with asymmetric information and discontinuous expected payoffs. Such results can be applied to concrete economies with asymmetric information and they will enable us to obtain competitive equilibrium/rational expectations equilibrium results with discontinuous expected payoffs. He and Yannelis (2014) considered discontinuous games with asymmetric information under ambiguity. They showed that if agents face ambiguity, then Reny's theorem can be generalized to an asymmetric information framework.

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[5] Instead, one can assume that $(T_i, \mathcal{T}_i, \lambda_i)$ is an atomless Loeb space/saturated space for each $i \in I$. The purification result for Bayesian games still holds, see Loeb and Sun (2006) and Wang and Zhang (2012).

[6] We thank an anonymous referee for suggesting us to add this remark.

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